Effect of Perturbed Orbital Motion on a Spinning Symmetrical Satellite

RALPH PRINGLE JR.*
Virginia Polytechnic Institute and State University, Blacksburg, Va.

This paper extends the "Thomson problem," motion of a symmetrical rigid body in an inverse-square central-force circular orbit under gravity-gradient torques, to the case of a precessing circular orbit under the gravitational attraction of an oblate primary. The averaging method is applied to the modal equations of motion. It is shown that as the spin rate increases, the deviations of the body symmetry axis from the orbit normal become large, due to resonance. In this large motion, for spin rate greater than a certain critical value, the spin axis ceases to remain near the orbit normal and describes a wandering path in inertial space.

I. Introduction

IN a classic paper, Thomson[1] analyzed the motion of a spinning symmetrical satellite in a circular orbit around a *point mass primary*. Specifically, he showed that an equilibrium orientation exists when the body symmetry axis points along the orbit angular momentum vector with spin only about the body symmetry axis. This equilibrium was shown to be stable for sufficiently large values of spin angular velocity. Later authors[2,3] clarified and corrected Thomson's stability diagram. Various other equilibrium attitudes were discussed in Pringle[4] and later in Likins[5] and Beletskii.[6] All of this work assumes a point mass primary and results in stability of the "Thomson problem" equilibrium for indefinitely large-spin angular velocity.

If the primary attracting body is *oblate*, as in the case of the Earth, the orbit plane (nodal angle) regresses with time and Thomson's model must be modified. For high spin rates, the body symmetry axis has a tendency to remain fixed in inertial space as the orbit plane rotates. Depending on the spin rate and the magnitude of the orbit plane nodal regression (i.e., the amount of oblateness), the spin axis will either follow the orbit angular momentum vector as it rotates or follow an inertial space-referenced path, perturbed by the primary gravity gradient effect. A recent paper by Cochrane[7] analyzes a similar physical problem for high-spin motion, using a different method.

This paper analyzes the Thomson problem with Earth oblateness effects included. The analysis of the motion for high ratios of spin angular rate to orbit mean motion is reduced to considering a nonlinear oscillator driven by a sinusoidal forcing function. Let W be the force-free precession frequency divided by mean motion and r be the ratio of symmetry axis moment of inertia to transverse axis moment of inertia. Normalize time, τ, so that one time unit equals 1 rad of orbit motion. The angular deviations θ_1 and θ_2 of the symmetry axis from the orbit's osculating angular momentum vector are described by the linearized differential equations

$$\begin{aligned}\ddot{\theta}_1+(W-1)\dot{\theta}_2+W\theta_1 &= -[\dot{n}_1+Wn_2]\\ \ddot{\theta}_2-(W-1)\dot{\theta}_1+[W+3(r-1)]\theta_2 &= -[\dot{n}_2-Wn_1]\end{aligned} \tag{1}$$

The angular rates n_1, n_2 about the in-plane axis system are proportional to the oblateness parameter, J_2, and the $\sin 2I$, where I is the orbit inclination above the equatorial plane. The rates n_1, n_2 are given by ($\tau = nt$)

$$n_1 = (-\Lambda/2)\cos\tau$$
$$n_2 = (3\Lambda/2)\sin\tau$$

Received November 27, 1972; revision received January 24, 1974.
Index category: Spacecraft Attitude Dynamics and Control.

* Associate Professor, Department of Engineering Science and Mechanics; presently Research Staff Member, Institute for Defense Analyses, Arlington, Va. Associate Fellow AIAA.

where Λ depends on orbital parameters. Thus the motions of θ_1, θ_2 are forced at orbital angular frequency.

When we assume motions θ_1, θ_2 vary at frequency ω as $e^{i\omega\tau}$, the characteristic equation of unforced motion is ($k = 1, 2$)

$$\omega_k^4-[W^2+3r-2]\omega_k^2+W[W+3(r-1)] = 0 \tag{2}$$

When $W > 5$, then the roots of (2) are approximated well by

$$\begin{aligned}\omega_1^2 &= 1+[3(r-1)(W-1)/W^2]\\ \omega_2^2 &= W^2[1+3(r-1)/W^2]\end{aligned} \tag{3}$$

For large W, the gravity-gradient effects diminish in importance and one natural frequency, ω_1, approaches the orbital mean motion as the other, ω_2, approaches the force-free precession frequency.

Thus the ω_1-mode is nearly resonant with the forcing frequency due to oblateness. The steady-state motion of θ_1 and θ_2 will tend to grow larger as W grows. The size of θ_1 and θ_2 in resonance dictates that a nonlinear analysis be performed. This analysis leads to jump phenomena in amplitude $(\theta_1^2+\theta_2^2)^{1/2}$. It will be shown that in the phase space of amplitude-phase of the ω_1-mode, two stable steady-state amplitudes exist for $W < W_c$, where W_c is some critical value. For $W > W_c$ there will be only one steady-state oscillation and it will be of large amplitude. This analysis will now be carried out in detail for the nonlinear case of $W > 10$ to determine the practical limits of Thomson-type equilibrium motion.

II. Equations of Motion

Let us consider a satellite whose center of mass moves in a perturbed circular orbit. The basic coordinate system, rotating with the orbiting mass center, is described by three unit vectors $\hat{1}$, $\hat{2}$, $\hat{3}$. The $\hat{1}$ unit vector is directed outward along the radius vector $\mathbf{R}^c$ from the primary to the satellite mass center; $\hat{3}$ is normal to the instantaneous osculating ellipse; and $\hat{2} = \hat{3}\times\hat{1}$. The body generalized coordinates are taken to be a set of Euler angles. Let $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ be a nonspinning set of unit vectors with $\hat{e}_3$ along the body symmetry axis, and define as follows a set of Euler angles describing $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ with respect to $\hat{1}$, $\hat{2}$, $\hat{3}$: rotate (right-handed) first about $\hat{1}$ by angle θ_1, then about $\hat{e}_2$ by angle θ_2; the spin rate about $\hat{e}_3$ is denoted $\dot{\psi}$, where ψ is the body rotation angle with respect to $\hat{e}_1$ and $\hat{e}_2$.

The angular velocity of the $\hat{1}$, $\hat{2}$, $\hat{3}$ frame with respect to inertial space is $\mathbf{n} = n_1\hat{1}+n_2\hat{2}+n_3\hat{3} = N_1\hat{e}_1+N_2\hat{e}_2+N_3\hat{e}_3$.

The kinetic energy T and the potential energy V for rotation of a body with principal axes along $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ and moments of inertia $I_1 = I_2 \neq I_3$ about these axes are, respectively[4]

$$T = (I_1/2)[(\Omega_1^2+\Omega_2^2)+r\Omega_3^2] \tag{4}$$

$$V = \tfrac{3}{2}(kI_1/|\mathbf{R}^c|^3)(r-1)\sin^2\theta_2+V_{J2} \tag{5}$$

where V_{J2} is the potential term due to Earth oblateness, $I_1 = I_2$ for a symmetrical body, and k is the gravitational constant. We have normalized the variables so that the orbital mean angular velocity $n = 1$, $r = I_3/I_1$, and Ω_1, Ω_2, Ω_3 are defined as

$$\begin{aligned}\Omega_1 &= \dot\theta_1 \cos\theta_2 + N_1\\ \Omega_2 &= \dot\theta_2 + N_2\\ \Omega_3 &= \dot\psi + \dot\theta_1 \sin\theta_2 + N_3\end{aligned} \qquad (6)$$

By definition, the momenta conjugate to θ_1, θ_2, ψ are

$$\begin{aligned}p_1 &= \Omega_1 \cos\theta_2 + p_\psi \sin\theta_2 = (\partial T/\partial\dot\theta_1)/I_1\\ p_2 &= \Omega_2 = (\partial T/\partial\dot\theta_2)/I_1\\ p_\psi &= r\Omega_3 \triangleq h = (\partial T/\partial\dot\psi)/I_1\end{aligned} \qquad (7)$$

The momentum p_ψ is a constant of the motion, since ψ does not appear in $L = T - V$, the Lagrangian. The Hamiltonian will be divided by I_1 and the momenta will also be so normalized. This produces the following formulation.

The equations of motion are canonical with

$$\begin{aligned}\dot p_1 &= -\partial\mathscr{H}/\partial\theta_1; & \dot\theta_1 &= \partial\mathscr{H}/\partial p_1\\ \dot p_2 &= -\partial\mathscr{H}/\partial\theta_2; & \dot\theta_2 &= \partial\mathscr{H}/\partial p_2\\ \dot p_\psi &= 0; & \dot\psi &= \partial\mathscr{H}/\partial p_\psi\end{aligned}$$

where the Hamiltonian

$$\mathscr{H} = \sum_{i=1}^{2} p_i\dot\theta_i + p_\psi\dot\psi - L/I_1$$

The vector $\mathbf{n}$ has components N_1, N_2, N_3 in $\hat e_1$, $\hat e_2$, $\hat e_3$ given by the expressions

$$\begin{aligned}N_1 &= n_1\cos\theta_2 + n_2\sin\theta_1\sin\theta_2 - n_3\cos\theta_1\sin\theta_2\\ N_2 &= n_2\cos\theta_1 + n_3\sin\theta_1\\ N_3 &= n_1\sin\theta_2 - n_2\sin\theta_1\cos\theta_2 + n_3\cos\theta_1\cos\theta_2\end{aligned}$$

The Hamiltonian for the motion in the θ_1, θ_2, ψ, p_1, p_2, p_ψ phase variables is from Eqs. (4) and (5) and

$$\mathscr{H} = \sum_{i=1}^{2} p_i\dot\theta_i + p_\psi\dot\psi - L/I_1$$

$$\mathscr{H} = \tfrac{1}{2}\{[(p_1 - p_\psi\sin\theta_2)/\cos\theta_2]^2 + p_2{}^2\} + (p_\psi{}^2/2r) - (N_1/\cos\theta_2)(p_1 - p_\psi\sin\theta_2) - N_2p_2 - N_3p_\psi + \tfrac{3}{2}(k/|\mathbf{R}^c|^3)(r-1)\sin^2\theta_2 + (V_{J2}/I_1) \qquad (9)$$

An alternate expression for $\mathscr{H}$ is derived from Eq. (9) by using Eqs. (6) and (7) and expressing $\mathscr{H}$ in terms of $\dot\theta_1$, $\dot\theta_2$ instead of p_1, p_2

$$\mathscr{H} = T_2{}^* + U \qquad (10)$$

where p_ψ is a constant and

$$\begin{aligned}T_2{}^* &= \tfrac{1}{2}[\dot\theta_1{}^2\cos^2\theta_2 + \dot\theta_2{}^2]\\ U &= V - \tfrac{1}{2}(N_1{}^2 + N_2{}^2) - p_\psi N_3\end{aligned}$$

U is a function of θ_1, θ_2 only and is called the dynamic potential.[4]

If we assume a circular orbit and include the first harmonic of the Earth's potential in our analysis, the triad $\hat 1$, $\hat 2$, $\hat 3$ has a rotational motion given approximately[8] by

$$\begin{aligned}n_1 &= -(\Lambda/2)\cos nt\\ n_2 &= (3\Lambda/2)\sin nt\\ n_3 &= n[1 + 3\bar J_2(1 - \tfrac{3}{2}\sin^2 I) - 3\bar J_2\sin^2 I\cos nt]\end{aligned} \qquad (11)$$

where $\bar J_2 = J_2(Re/a)^2$, a, Ω, R_E, I, n are, respectively, coefficient of oblateness, semimajor axis, argument of the ascending node, Earth radius, inclination, and mean angular motion. We have defined $\Lambda = \tfrac{3}{2}\bar J_2\cos I\sin I$ and carried the expansions up to order $\bar J_2$. The mean motion is related to $\mathbf{R}^c$ by $n^2 = k/|\mathbf{R}^c|^3$. Time $\tau = nt$ is zero at passage of the ascending node.

III. Equilibrium Points

From Lagrange's equations, it can be shown that a point of equilibrium such that $\ddot\theta_1 = \ddot\theta_2 = \dot\theta_1 = \dot\theta_2 = 0$ exists if

$$\partial U/\partial\theta_1 = \partial U/\partial\theta_2 = 0 \qquad (12)$$

Let $J_2 = 0$. Then $\mathscr{H}$ is not a function of t and $\dot{\mathscr{H}} = 0$ by a well-known theorem. $\mathscr{H}$ is therefore a constant of the motion. We find the equilibrium points by Eq. (12). We have ($h \equiv p_\psi$)

$$U = \tfrac{3}{2}(r-1)\sin^2\theta_2 - h\cos\theta_1\cos\theta_2 + \tfrac{1}{2}(\cos^2\theta_1\cos^2\theta_2) \qquad (13)$$

Condition (12) becomes

$$\partial U/\partial\theta_1 = [h - \cos\theta_1\cos\theta_2]\sin\theta_1\cos\theta_2 = 0 \qquad (14)$$

$$\partial U/\partial\theta_2 = [3(r-1)\cos\theta_2 + h\cos\theta_1 - \cos^2\theta_1\cos\theta_2]\sin\theta_2 = 0 \qquad (15)$$

We have two major cases where Eqs. (14) and (15) are satisfied:

Case I:

$$\theta_1 = \theta_2 = 0$$

Case II:

a) $\theta_1 = 0$; $\cos\theta_2 = h/(4-3r)$

b) $\theta_2 = 0$; $\cos\theta_1 = +h$

In an earlier paper,[4] this writer analyzed the problem of bounds on the motion about the abovementioned equilibrium points by using curves of constant $U(\theta_1, \theta_2)$ plotted on a unit sphere. The relation $\mathscr{H} = T_2{}^* + U$ leads to an inequality $U \leqq \mathscr{H}_o$, where $\mathscr{H}_o$ is the initial value of $\mathscr{H}$. Of course, since $\dot{\mathscr{H}} = 0$, $\mathscr{H} = \mathscr{H}_o$ for all t. If there is a local minimum of U as a function of θ_1, θ_2 at a certain equilibrium point, P_E, of the system, then curves of constant U in θ_1, θ_2-space surround P_E. If a certain set of initial conditions is given, these fix $\mathscr{H}_o$ and determine a point P_I $(\theta_{10}, \theta_{20})$. If P_I lies within the closed curve, $U = \mathscr{H}_o$. Thus, curves of constant U are bounds on the motion if U is minimum at points P_E.

To classify the various equilibrium points as a function of h and r, we plot certain "curves of bifurcation."[4] These separate certain qualitative behavior of the U-curves. The corresponding curves of constant U are plotted on the surface of a unit sphere in Beletskii.[6] Beletskii drew the figures in a more descriptive manner but patterned his analysis after Ref. 4.

The motion about points of equilibrium, P_E, where U is a *minimum*, possesses an upper bound on the motion and is *stable* in the sense of Liapunov. If the equations of motion are linearized around P_E, we obtain solutions of the form $e^{\neq i\omega_j t}$ $(j = 1,2)$, where the ω_j $(j = 1,2)$ are natural frequencies of the stable motion.

The three types of equilibrium points are discussed in detail in Refs. 4 and 5. Here we discuss only Thomson's equilibrium of Case I to prepare the way for the perturbation analysis to follow.

In the case $(\theta_1 = \theta_2 = 0)$, the motion exhibits bounded oscillations for regions where $|h| > 4 - 3r$, $|h| > 1$, $|h| > 3r - 4$ as is shown in Ref. 4.

IV. Perturbation Analysis of Resonance Motion

In this section a perturbation analysis of various resonance types is carried out for bounded motions of Case I, the case of equilibrium with spin normal to the orbit plane $(\theta_1 = \theta_2 = 0)$. We use equations of motion in canonical form of Eqs. (8) and (9). First we must expand $\mathscr{H}$ in powers of θ_1, θ_2, p_1, p_2, $\bar J_2$. Let $\mathscr{R}$ denote the order of magnitude of θ_1, θ_2, p_1, p_2. We carry our expansions up to terms of order $\mathscr{R}^4$, $\bar J_2\mathscr{R}$, treating $0(\mathscr{R}) \sim 0.1$, $0(\bar J_2) \sim 0.001$. This gives a realistic result for small but finite oscillations. Note that n_1, n_2, n_3 are of order $\bar J_2$. The Hamiltonian is written as $\mathscr{H} = \mathscr{H}_2 + \mathscr{H}_4$, where

$$\mathscr{H}_2 = \tfrac{1}{2}[(p_1 - h\theta_2)^2 + p_2{}^2] + [-\theta_1 p_2 + \theta_2 p_1 + (h/2)(\theta_1{}^2 - \theta_2{}^2)] + \tfrac{3}{2}(r-1)\theta_2{}^2 \qquad (16)$$

$$\mathscr{H}_4 = \tfrac{1}{2}(p_1 - h\theta_2)^2\theta_2{}^2 + (h/6)\theta_2{}^3(p_1 - h\theta_2) - (h/24)(\theta_1{}^4 + \theta_2{}^4 + 6\theta_1{}^2\theta_2{}^2) - \frac{r-1}{2}\theta_2{}^4 + \theta_2(p_1 - h\theta_2) \times \left[\frac{\theta_2{}^2}{3} - \frac{\theta_1{}^2}{2}\right] + \frac{h\theta_2{}^4}{6} + \frac{p_2\theta_1{}^3}{6} - n_1p_1 - n_2(p_2 - h\theta_1) + V_{J2} + \frac{(n_3 - n)h}{2}(\theta_1{}^2 - \theta_2{}^2) \qquad (17)$$

We now make a canonical transformation of the p_i, θ_i phase variables into α_i, β_i phase variables.

$$\begin{bmatrix}\theta_1\\ \theta_2\\ p_1\\ p_2\end{bmatrix} = G\begin{bmatrix}(2\alpha_1/\omega_1)^{1/2}\sin(\omega_1 t+\beta_1)\\ (2\alpha_2/\omega_2)^{1/2}\sin(\omega_2 t+\beta_2)\\ (2\alpha_1\omega_1)^{1/2}\cos(\omega_1 t+\beta_1)\\ (2\alpha_2\omega_2)^{1/2}\cos(\omega_2 t+\beta_2)\end{bmatrix} = b_z \quad (18)$$

where G is a 4×4 matrix with coefficients as follows for $n = 1, 2$:

$$\begin{aligned} G_{1,n} &= 2a_n & G_{1,n+2} &= 0\\ G_{2,n} &= 0 & G_{2,n+2} &= 2k_{2n}a_n/\omega_n\\ G_{3,n} &= 0 & G_{3,n+2} &= 2k_{3n}a_n/\omega_n\\ G_{4,n} &= 2k_{4n}a_n & G_{4,n+2} &= 0 \end{aligned} \quad (19)$$

The ω_n $(n = 1, 2)$ are roots of the characteristic equation $(\omega_1 < \omega_2)$

$$0 = \omega_n^4 - ((3r-2)+(h-1)^2)\omega_n^2 + (h-1)[(h+3r-4)] \quad (20)$$

where Eq. (20) coincides with Eq. (2) when $W \triangleq h-1$. The parameters k_{2n}, k_{3n}, k_{4n}, a_n are defined

$$a_n = \left[\frac{\omega_n/4}{k_{3n}-k_{2n}k_{4n}}\right]^{1/2}$$

$$k_{2n} = \frac{[\omega_n^2-(h-1)^2]}{\omega_n(\omega_n^2+h-1+a)}$$

$$k_{3n} = \frac{[\omega_n^2 h + a(h-1)]}{\omega_n(\omega_n^2+h-1+a)}$$

$$k_{4n} = \frac{h^2-h+a}{(\omega_n^2+h-1+a)}$$

where

$$a = -[h^2-h+3(r-1)]$$

The reduction to normal modes for a canonical system is given in Whittaker[9] and employed in Ref. 10.

The expression for $\mathscr{H}$ with variables z_i is $\mathscr{H} = \frac{1}{2}(z_3^2 + \omega_1^2 z_1^2) + \frac{1}{2}(z_4^2 + \omega_2^2 z_2^2) + \mathscr{H}_4 + \cdots$. The equations of motion in the new α, β variables are of the form

$$\begin{aligned}\dot{\alpha}_i &= -\partial\mathscr{H}'/\partial\beta_i\\ \dot{\beta}_i &= \partial\mathscr{H}'/\partial\alpha_i\end{aligned} \quad (21)$$

where $\mathscr{H}' = \mathscr{H}_4 + \cdots$ with the transformation (19) substituted into $\mathscr{H}$ given in Eq. (17) up to $0(\mathscr{R}^4)$. The new Hamiltonian $\mathscr{H}'$ contains three types of terms: 1) terms quadratic and quartic in $\alpha_i^{1/2}$, 2) trigonometric terms with arguments whose time derivatives are of order 1 (short-period terms), and 3) (near resonance lines) trigonometric terms with arguments whose derivatives are small. These latter terms are called "long-period terms." If Eqs. (21) are integrated to find a first-order approximation with (α, β) held constant in the integration, we get small divisors arising from long-period terms. We eliminate these long-period terms using the method of averaging to obtain long-period changes in (α, β).

V. Effect of $J_2 \neq 0$

The motion of a rather rapidly spinning (h large) body can be seen to be stable if $|h| > 4$ for any r. Moreover, as h increases, $\omega_1 \to 1.0$. This can be seen from Eq. (20). In fact, a perturbation analysis of Eq. (20) for $h > 6$ gives approximately

$$\begin{aligned}\omega_1^2 &= 1 + [3(r-1)(W-1)/W^2]\\ \omega_2^2 &= W^2[1+3(r-1)/W^2\end{aligned} \quad (22)$$

where $W \triangleq h-1$ has been defined. From this point we restrict ourselves to the case $h > 0$. From physical arguments we can see that for $W \to \infty$, the gravity torque has diminishingly small effect and the motion approaches that of a free rigid body. That is precisely what Eq. (22) expresses. The limiting case $\omega_1 \to 1$ represents the motion of a body pointing fixed in inertial space with the coordinate system $(\hat{1}, \hat{2}, \hat{3})$ rotating at frequency 1 (the orbit mean angular motion). W is the free space precession frequency corrected for coordinate system rotation at mean orbital angular rate n $(=1)$.

Now when W is large for an unperturbed circular orbit, the motion is bounded and stable. When $J_2 \neq 0$, the orbit plane moves secularly (the node regresses). For some value of W, we must get a qualitative change in motion. That is, as W increases, the motion of $\hat{e}_3$ will cease following the $\hat{3}$ vector (orbit normal) and start moving as if pointing toward a point in inertial space. The condition $J_2 \neq 0$ changes the stability conclusions of Eq. (20) for *some* critical value of W.

The disturbing effects of $J_2 \neq 0$ are felt in the nonzero values of n_1, n_2. That is, $\hat{1}$, $\hat{2}$, $\hat{3}$ move with angular velocities about all three axes instead of just about $\hat{3}$. The expressions for n_1, n_2 are found in Eq. (11) to be sinusoidal of frequency $n = 1$. From Eq. (17), we see that n_1, n_2 have coefficients linear in p_1, p_2, θ_1 and thus they are forcing functions to the equations of motion in Eq. (8). Since the forcing terms due to $J_2 \neq 0$ drive the system at a near resonance $\omega_1 \cong 1$, we see that the mathematics confirms our physical speculations of the previous paragraphs. The forced motion will grow due to $J_2 \neq 0$.

The potential energy term V_{J2} represents an attitude-dependent influence of order J_2. For $W > 10$, the perturbing influence on $\mathscr{H}_4$ of orbit rotation is of order $J_2 h$, which is one order of magnitude larger than the term V_{J2}. We shall neglect the effects of V_{J2} for the case under study here.

To begin the perturbation analysis, we refer to Ref. 10 where similar analyses were carried out. The resonant mode $\omega_1 \simeq 1$ contributes long-period terms to $\mathscr{H}$. The long-period modal behavior is determined by averaging $\mathscr{H}$ to obtain $\bar{\mathscr{H}}^*$, then integrating the long-period equations for $\bar{\mathscr{H}}^*$.† First, we substitute Eqs. (18) into (17) and average to obtain $\bar{\mathscr{H}}^*$. The new averaged variables are (α_i^*, β_1^*)

$$\bar{\mathscr{H}}^* = \Delta_1\alpha_1^* + \Delta_2\alpha_1^{*2} + \Delta_3(\alpha_1^*)^{1/2}\cos\beta_1^* \quad (23)$$

where $\beta_1^* = \beta_1 + (\omega_1 - 1)\tau$, $\alpha_1^* = \alpha_1$, $\bar{\mathscr{H}}^* = \bar{\mathscr{H}}_4 + (\omega_1-1)\alpha_1^*$. The parameters are to order $1/W^2$ $(W = h-1)$ as follows:

$$\Delta_1 = 3(r-1)(W-1)/2W^2 > 0$$

$$\Delta_2 = -[(W+1)-3(r-1)/16W^2] < 0$$

$$\Delta_3 = (2)^{1/2}(W+1)\Lambda/W^{1/2} > 0$$

where Λ was defined previously as $\Lambda = \frac{3}{2}J_2\sin I\cos I$. The contributions of the long-period mode ω_1 to θ_1 and θ_2 are given by

$$\begin{aligned}\theta_1 &= (\omega_1/W^{1/2})[1-(1/2W)](2\alpha_1^*)^{1/2}\sin(\beta_1^*+\tau)\\ \theta_2 &= (1/\omega_1 W^{1/2})[1-(1/2W)](2\alpha_1^*)^{1/2}\cos(\beta_1^*+\tau)\end{aligned} \quad (24)$$

for $W \gg 1$.

VI. Long-Period Behavior

Note that Eq. (24) may be rewritten as

$$\begin{aligned}\theta_1 &= (\omega_1/W^{1/2})[1-(1/2W)][\xi\cos\tau + \eta\sin\tau]\\ \theta_2 &= (1/\omega_1 W^{1/2})[1-(1/2W)[-\xi\sin\tau + \eta\cos\tau]\end{aligned} \quad (25)$$

where a new canonical transformation has been defined as follows:

$$\begin{aligned}\xi &= (2\alpha_1^*)^{1/2}\sin\beta_1^*\\ \eta &= (2\alpha_1^*)^{1/2}\cos\beta_1^*\end{aligned} \quad (26)$$

with

$$\bar{\mathscr{H}}^* = (\Delta_1/2)(\xi^2+\eta^2) + (\Delta_2/4)(\xi^2+\eta^2)^2 + [\Delta_3/(2)^{1/2}]\eta \quad (27)$$

The motion of (ξ, η) obeys the canonical equations

$$\begin{aligned}\dot{\xi} &= +\partial\bar{\mathscr{H}}^*/\partial\eta = \Delta_1\eta + \Delta_2(\xi^2+\eta^2)\eta + \Delta_3/(2)^{1/2}\\ \dot{\eta} &= -\partial\bar{\mathscr{H}}^*/\partial\xi = -\Delta_1\xi - \Delta_2(\xi^2+\eta^2)\xi\end{aligned} \quad (28)$$

The motion in the (ξ, η) phase plane is characterized by the equilibrium points at $\dot{\xi} = \dot{\eta} = 0$. These conditions are as follows:

$$\begin{aligned}&[\Delta_1 + \Delta_2(\xi^2+\eta^2)]\eta + \Delta_3/(2)^{1/2} = 0\\ &[\Delta_1 + \Delta_2(\xi^2+\eta^2)]\xi = 0\end{aligned} \quad (29)$$

† The more rigorous Lie transformation method of Hori[11] and Deprit[12] gives the same result for long-period motion to first order.

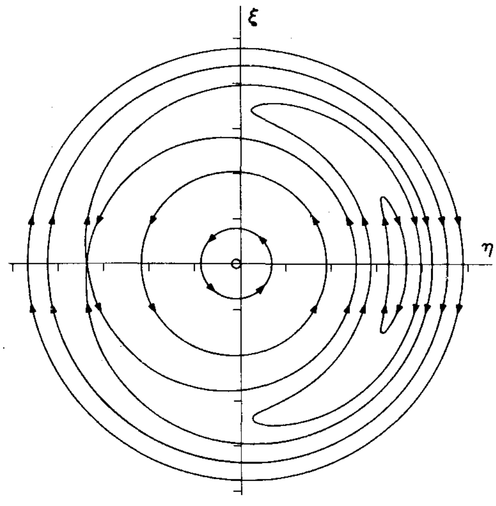

Fig. 1 Integral curves in $\xi-\eta$ plane for $W = 20$.

There are two possible solutions to the second condition: either 1) $\xi = 0$, or 2) $[\Delta_1 + \Delta_2(\xi^2+\eta^2)] = 0$. If (1) is true, then from the first conditions ($\dot{\xi} = 0$)

$$(\Delta_1 + \Delta_2\eta^2)\eta + \Delta_3/(2)^{1/2} = 0 \tag{30}$$

is a cubic equation in η. If (2) is true, then $\Delta_3 = 0$ (or $J_2 = 0$), which is not an interesting case for our physical problem. We proceed to investigate Eq. (30) for the equilibrium points in the (ξ, η) phase plane.

From Eq. (30) we see that there exist in the real domain either three solutions or one solution for η. If we construct the curve $f(\eta) = \Delta_1\eta + \Delta_2\eta^3$, we see that roots of Eq. (30) correspond to values of Δ_3 such that $f(\eta) = -\Delta_3/(2)^{1/2}$. For small enough $|\Delta_3|$ and for the case $(\gamma > 0)$ $\Delta_1 > 0$, $\Delta_2 < 0$, $\Delta_3 > 0$, there are three roots. For very large Δ_3, there is one root. The separating value, $\Delta_3 = \Delta_{3\max}$, is defined as the value of η for which $f(\eta)$ is a maximum. Thus $f'(\eta) = 0$ at this point and it may be shown that

$$\Delta_{3\max} = 4(2)^{1/2}\Delta_1^{3/2}/(3)^{3/2}|\Delta_2|^{1/2}$$

at this point. This condition defines a critical value of W, W_c, above which only one (large) real equilibrium value of η exists, and below which there are three such values of η. Approximately, this critical W is

$$W_c \cong 2^{5/3}(r-1)/\Lambda^{2/3} \quad (r > 1.0)$$

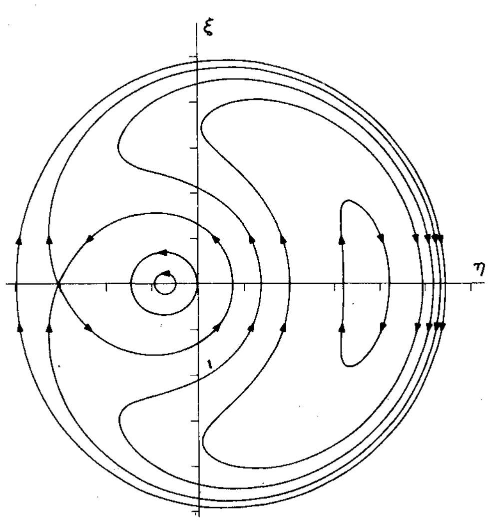

Fig. 2 Integral curves in $\xi-\eta$ plane for $W = 75$.

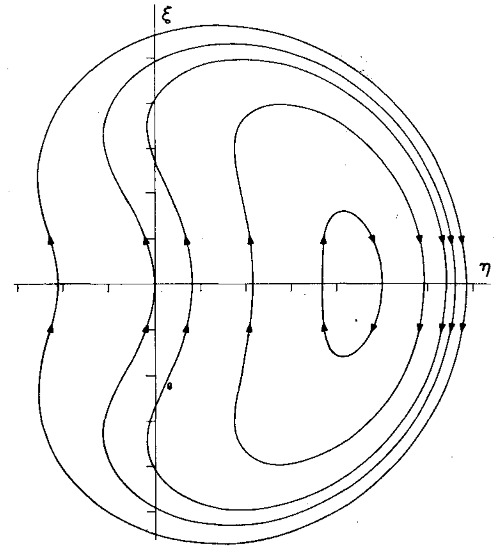

Fig. 3 Integral curves in $\xi-\eta$ plane for $W = 192.3$ (critical case).

For spin rate Ω_3 and $W = h-1 = r\Omega_3 - 1$, then the critical spin speed

$$\Omega_{3c} \cong 2^{5/3}(1-1/r)/\Lambda^{2/3}$$

corresponds to W_c. As an example let $n = 0.001$, $\Lambda = 0.001$, $r = 1.5$; then $W_c \cong 165$, $\Omega_c = 110$, and the critical angular velocity is approximately 1 rpm.

For the case ($r = 1.5$, $\Lambda = 0.001$), we have studied three values of $W = 20$, 75, 192.3. The (ξ, η) phase planes for these cases appear in Figs. 1–3.‡ The case $W = 192.3$ is the true critical one above which there exists but one equilibrium point. Note that for $W = 20$ or 75 and for large enough ξ and η, there may be motions for which the average value of η is large. Since the (ξ, η) plane is equivalent to viewing θ_1 and θ_2 in inertial space [see Eq. (25)], the motion with large average values of η is "locked onto inertial space." This is easily possible for $W = 75$, even for relatively small initial η. For $W = 192.3$ and above nearly all motions have a bias to the positive values of average η. That is, above $W_c = 192.3$ for this case, the motions remain fixed in inertial space over a single orbit but still follow the very slow orbit plane motion.

VII. Numerical Check

The complete nonlinear equations of motion still using the same orbital motion model were numerically integrated for the case $W = 75$, $r = 1.5$, $\Lambda = 0.001$. Plots were made of $(\bar{\xi}, \bar{\eta})$ where

$$\bar{\xi} = \bar{\theta}_1 \cos\tau - \bar{\theta}_2 \sin\tau$$
$$\bar{\eta} = \bar{\theta}_1 \sin\tau + \bar{\theta}_2 \cos\tau \tag{31}$$

where

$$\bar{\theta}_1 = \frac{W^{1/2}\theta_1}{\omega_1(1-1/2W)}; \qquad \bar{\theta}_2 = \frac{\omega_1 W^{1/2}\theta_2}{(1-1/2W)}$$

These runs verified several motions shown in Fig. 2, i.e., the (ξ, η) equilibrium point motion and the motion beginning at $\xi = 0$, $\eta = 1.8$. The numerical integrations were expensive in computer time. The rapid oscillation of mode $\omega_2 \cong W = 75$ determined step size, whereas phase plane motion was very slow of order 0.01 rad/time unit. Therefore the expense of numerical integration of such problems is a good argument for averaging methods.

‡ Note that the tick marks on ξ, η axes are spaced by units of one in ξ, η.

References

[1] Thomson, W. T., "Spin Stabilization of Attitude Against Gravity Torque," *Journal of the Astronautical Sciences,* Vol. 9, 1962, pp. 31–33.

[2] Kane, T. R. et al., "Letter to the Editor," *Journal of the Astronautical Sciences,* Vol. 9, No. 4, 1962, pp. 108–109.

[3] Chernous'ko, F. L., "On the Stability of Regular Precession of a Satellite," *Applied Mathematics and Mechanics,* Vol. 28, No. 1, 1964, pp. 181–184.

[4] Pringle, R., Jr., "Bounds on the Librations of a Symmetrical Satellite," *AIAA Journal,* Vol. 2, No. 3, May 1964, pp. 908–912.

[5] Likins, P. W., "Stability of a Symmetrical Satellite in Attitudes Fixed in an Orbiting Reference Frame," *Journal of the Astronautical Sciences,* Vol. 12, 1965, pp. 18–24.

[6] Beletskii, V. N., "Motion of an Artificial Satellite about its Center of Mass," TT F-429, 1966, NASA.

[7] Cochrane, J. E., "Effects of Gravity-Gradient Torque on the Rotational Motion of a Triaxial Satellite in a Precessing Elliptic Orbit," *Celestial Mechanics,* Vol. 6, No. 2, Sept. 1972, pp. 126–150.

[8] Claus, A. J. and Lubowe, A. G., "A High Accuracy Perturbation Method with Direct Application to Communication Satellite Orbit Prediction," *Astronautical Acta,* Vol. 9, 1963, pp. 275–301.

[9] Whittaker, E. T., *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies,* Cambridge University Press, New York, 1904, pp. 426–429.

[10] Breakwell, J. V. and Pringle R., Jr., "Nonlinear Resonance Affecting Gravity-Gradient Stability," *Astrodynamics, Proceedings of the 16th International Astronautical Federation Congress (1965),* Gauthier-Villars, Paris, 1966, pp. 305–325.

[11] Hori, G., "Theory of General Perturbations with Unspecified Canonical Variables," *Publications of the Astronautical Society of Japan,* Vol. 19, 1966, pp. 229–241.

[12] Deprit, A., "Canonical Transformations Depending on a Small Parameter," *Celestial Mechanics,* Vol. 1, No. 1, 1969, pp. 12–30.